A Derivation of Amortization — *Bret D. Whissel*

This is my derivation of the formula for amortization from which this program is developed. Observe the following variables:

P	The principal borrowed
I	The total interest paid
N	The number of payments
i	The fractional (periodic) interest rate
P_j	The principal part of payment j
I_j	The interest part of payment j
B	A final balloon payment
x	The regular payment

Assuming that all payments (excluding a final balloon payment) are the same amount, a payment consists of its interest part and its principal part:

$$x = I_j + P_j \tag{1}$$

$$\begin{aligned}
I_1 &= iP & P_1 &= x - I_1 \\
I_2 &= i(P - P_1) & P_2 &= x - I_2 \\
I_3 &= i(P - P_1 - P_2) & P_3 &= x - I_3 \\
I_4 &= i(P - P_1 - P_2 - P_3) & P_4 &= x - I_4, \text{ etc.}
\end{aligned}$$

This payment schedule assumes that the current payment x includes interest on all of the remaining principal, including principal which is part of the current payment. Therefore, the first payment includes an interest payment on all of the borrowed principal. The P_j's may be re-written into a recurrence relation:

$$P_1 = x - iP$$

$$\begin{aligned}
P_2 &= x - i(P - P_1) \\
&= x - i\left[P - (x - iP)\right] \\
&= x - iP + ix - i^2 P \\
&= (x - iP)(1 + i)
\end{aligned}$$

$$\begin{aligned}
P_3 &= x - i(P - P_1 - P_2) \\
&= x - i\left[P - (x - iP) - (x - iP + ix - i^2 P)\right] \\
&= x + 2ix + i^2 x - iP - 2i^2 P - i^3 P \\
&= x(1 + 2i + i^2) - iP(1 + 2i + i^2) \\
&= x(1 + i)^2 - iP(1 + i)^2 \\
&= (x - iP)(1 + i)^2
\end{aligned}$$

In general, we will find that

$$P_j = (x - iP)(1 + i)^{j-1}. \tag{2}$$

If there is to be a balloon payment, then the final payment will consist of the final principal payment P_f and interest on that principal iP_f so that $B = P_f + iP_f$. Rewriting P_f in terms of B gives $P_f = B/(1+i)$.

In order to solve for x, one more statement must be made.

$$B + Nx = P + \sum_{j=1}^{N} I_j + i\left(\frac{B}{1+i}\right), \tag{3}$$

or in English, the sum of all the payments (left side) is equal to the principal borrowed plus all of the interest paid with regular payments plus interest paid on the balloon payment (right side). Note that if there will be no balloon payment ($B = 0$), then the B terms drop out.

OK, now for the hard part. We replace I_j of equation (3) using the relationship given by equation (1) and then substitute the recurrence identity of equation (2):

$$B - \frac{iB}{1+i} + Nx = P + \sum_{j=1}^{N} \left[x - (x - iP)(1 + i)^{j-1}\right]$$

$$B - \frac{iB}{1+i} + Nx = P + Nx - (x - iP)\sum_{j=1}^{N}(1 + i)^{j-1}$$

$$P - B\left(1 - \frac{i}{1+i}\right) = (x - iP)\sum_{j=1}^{N}(1 + i)^{j-1}$$

We can rewrite the limits of the summation now:

$$P - B\left(1 - \frac{i}{1+i}\right) = (x - iP)\sum_{j=0}^{N-1}(1 + i)^{j}$$

and finally solve for x:

$$x = \frac{P - B\left(1 - \frac{i}{1+i}\right)}{\sum_{j=0}^{N-1}(1 + i)^j} + iP$$

Thanks to equation 3.2.13 on page 227 of *Discrete Mathematics for Computer Scientists* by Joe Mott, Abe Kandel, and Ted Baker, I can rewrite the summation in a non-iterative form so that:

$$x = \left[P - B\left(1 - \frac{i}{1+i}\right)\right]\frac{1 - (1 + i)}{1 - (1 + i)^N} + iP$$

At last, we have the analytic solution! *Quod erat demonstrandum* ("That which was to be shown"), otherwise known as Q.E.D. —BDW

HELP! Can anyone solve this equation for i? When i is the unknown, I'm currently using an iterative method to converge on the periodic interest rate.